Exam. Code : 211003
Subject Code : 4868

M.Sc. Mathematics 3rd Semester

NUMBER THEORY

Paper—MATH–586

Time Allowed—3 Hours] [Maximum Marks—100

Note :— Candidates are required to attempt **FIVE** questions, selecting at least **ONE** question from each section. The **fifth** question may be attempted from any section.

SECTION—A

1. (a) Obtain three consecutive integers, each having a square factor. 10

   (b) State and prove Wolstenholme's theorem. 10

2. (a) For Fermat numbers F_m and F_n, $m > n \geq 0$, prove that gcd $(F_m, F_n) = 1$. 10

   (b) Let r be a primitive root of integer n. Find the necessary and sufficient condition for r^k to primitive root of the integer n. 10

SECTION—B

3. (a) If r is a primitive root of the odd prime p, verify that

$$\text{ind}_r(-1) = \text{ind}_r(p-1) = \frac{1}{2}(p-1).$$ 10

   (b) Find all quadratic residues of 17. 10

4. State and prove Quadratic Reciprocity Law. 20

SECTION—C

5. (a) Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square. 10

   (b) State and prove Möbius Inversion Formula. 10

6. Find all solutions (a, b, c) of $x^2 + y^2 = z^2$ with gcd(a, b, c) = 1, a even and a > 0, b > 0 and c > 0. Further, prove that ab is divisible by 12 and $60 \mid abc$. 20

SECTION—D

7. (a) Prove that the value of any infinite continued fraction is an irrational number. 10

   (b) Let x be an arbitrary irrational number. If the rational number a/b, where $b \geq 1$ and gcd(a, b) = 1, satisfies $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$ then prove that a/b is one of the convergents in the continued fraction representation of x. 10

8. (a) Let x_1, y_1 be the fundamental solution of $x^2 - dy^2 = 1$. Then prove that every pair of integers x_n, y_n defined by the condition

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n n = 1, 2, 3, ...$$

   is also a positive solution. 10

   (b) Exhibit the solution of the equation $x^2 - 41y^2 = -1$. 10