

Exam. Code : 211003

Subject Code : 3856

M.Sc. (Mathematics) 3<sup>rd</sup> Semester

MATH-586 : NUMBER THEORY

Time Allowed—3 Hours]

[Maximum Marks—100

Note :— Attempt any TWO questions from each unit. All questions carry equal marks.

### UNIT—I

1. (a) Solve  $x \equiv 1 \pmod{3}$ ,  $x \equiv 2 \pmod{5}$ ,  $x \equiv 3 \pmod{7}$ .  
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- (b) Prove that the Fermat number  $F_5$  is divisible by 641.  
5
2. State and prove Wolsten-Holme's Theorem. 10
3. (a) If the integer  $a$  has order  $k$  modulo  $n$ , then prove that for  $h > 0$ , the order of  $a^h$  is  $k / \gcd(h, k)$  modulo  $n$ .  
5
- (b) If  $r$  is a primitive root of odd prime  $p$ , then prove that  $r^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ .  
5
4. Prove that an integer  $n > 1$  has a primitive root if and only if  $n = 2, 4, p^k$  or  $2p^k$ ,  $p$  odd prime. 10

### UNIT—II

5. (a) Let  $n$  be an integer possessing a primitive root and let  $\gcd(a, n) = 1$ . Prove that the congruence  $x^k \equiv a \pmod{n}$  has a solution if and only if  $a^{\phi(n)/d} \equiv 1 \pmod{n}$ .  
5
- (b) If  $r$  is a primitive root of the odd prime  $p$ , then prove that  $\text{ind}_r(-1) = \text{ind}_r(p-1) = \frac{p-1}{2}$ .  
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6. (a) Let  $r$  be a quadratic residue of odd prime  $p$  and  $ab \equiv r \pmod{p}$ . Prove that  $a$  and  $b$  both are quadratic residues of  $p$  or both are quadratic non-residues of  $p$ .  
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- (b) For a primitive root  $r$  of odd prime  $p$ , prove that the product of quadratic residues of  $p$  is congruent to  $r^{(p^2-1)/4}$  modulo  $p$ .  
5
7. State and prove Gauss Lemma. 10
8. (a) Prove that there are infinitely many primes of the form  $5k - 1$ .  
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- (b) For an odd prime  $p$ , show that  $\sum_{a=1}^{p-2} \left( \frac{a(a+1)}{p} \right) = -1$ .  
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### UNIT—III

9. (a) Find the form of all positive integers  $n$  such that  $\tau(n) = 10$ . What is the smallest positive integer  $n$  for which  $\tau(n) = 10$ ?  
5
- (b) Find  $\sum_{d|n} \mu(d)$  for each positive integer  $n \geq 1$ . 5

10. (a) Let  $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$  be the prime factorization of  $n$ . Prove that  $\sum_{d|n} \mu(d) \sigma(d) = (-1)^r p_1 p_2 \dots p_r$ .

5

- (b) For a perfect number  $n$ , prove that  $\sum_{d|n} \frac{1}{d} = 2$ .

5

11. Prove that if for  $k > 1$ ,  $2^k - 1$  is prime, then  $2^{k-1}(2^k - 1)$  is perfect and every even perfect number is of this form.
12. Prove that an odd prime  $p$  is expressible as sum of two squares if and only if  $p \equiv 1 \pmod{4}$ .

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#### UNIT—IV

13. Prove that a positive integer  $n$  is expressible as sum of two squares if and only if each of its prime factors of the form  $4k + 3$  occurs to an even power.
14. Prove that any prime can be written as sum of four squares.
15. State and prove Hurwitz Theorem.

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16. (a) For two successive terms  $\frac{a_1}{b_1}$  and  $\frac{a_2}{b_2}$  of  $F_n$ , prove that  $b_1 + b_2 > n$ .

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- (b) If  $n$  is a positive integer and  $x$  is a real number, then prove that there is a fraction  $\frac{a}{b}$  such that

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b(n+1)}.$$

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#### UNIT—V

17. (a) If  $\frac{p_n}{q_n}$  is the  $n^{\text{th}}$  convergent of the continued fraction

$$\langle a_0, a_1, \dots, a_n \rangle, \text{ show that } \langle a_n, a_{n-1}, \dots, a_1 \rangle = \frac{q_n}{q_{n-1}}.$$

5

- (b) Evaluate  $\langle -3, 2, 4, 5, 2 \rangle$ .

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18. (a) Expand  $\frac{5 + \sqrt{37}}{4}$  as continued fraction.

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- (b) Prove that the even convergents of infinite continued fraction forms a strictly increasing sequence.

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19. (a) Prove that if  $p$  and  $q$  are positive integers such that  $p^2 - dq^2 = 1$ , then  $\frac{p}{q}$  is a convergent of the continued

fraction expansion of  $\sqrt{d}$ .

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- (b) Show that  $x^2 - dy^2 = -1$  has no solution if  $d \equiv 3 \pmod{4}$ .

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20. Prove that if  $(x_1, y_1)$  is the fundamental solution of  $x^2 - dy^2 = 1$ , then all positive solutions are given by  $(x_n, y_n)$ , where  $x_n, y_n$  are the integers such that :

$$x_n + y_n \sqrt{d} = (x_1 + y_1 \sqrt{d})^n, n = 1, 2, 3, \dots$$

Further find the fundamental solution of  $x^2 - 48y^2 = 1$ .

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